Matrix exponentials

Let A be a complex square matrix, put

$$Q(z) = (z - a_1)^{\alpha_1+1} \cdots (z - a_k)^{\alpha_k+1},$$

with the a_p distinct and the α_p nonnegative integers, assume $Q(A) = 0$, set

$$Q_p(z) := \frac{Q(z)}{(z - a_p)^{\alpha_p+1}} \quad ,$$

let $b_{p,n}$ be the n-th Taylor coefficient of $1/Q_p(z)$ at $z = a_p$, let f be an entire function, and let $P(f(z), z) \in \mathbb{C}[z]$ be $Q(z)$ times the singular part of $f(z)/Q(z)$.

Theorem. *We have*

- $$b_{p,n} = (-1)^n \sum_{\substack{\beta_p=0 \\ |\beta|=n}} \prod_{\substack{j=1,\dots,k \\ j\neq p}} \binom{\alpha_j + \beta_j}{\alpha_j} \frac{1}{(a_p - a_j)^{\alpha_j+1+\beta_j}}$$

  where β runs over $\mathbb{N}^k$ and $|\beta| := \beta_1 + \cdots + \beta_k$,

- $$P(f(z), z) = \sum_{p=1}^{k} \sum_{q=0}^{\alpha_p} \sum_{j=0}^{q} \frac{f^{(j)}(a_p)}{j!} b_{p,q-j} (z - a_p)^q Q_p(z),$$

- $f(A) = P(f(z), A)$.

Let Q be as above and d its degree, and define $g_j : \mathbb{R} \to \mathbb{C}$ for $0 \le j < d$ by

$$P(e^{tz}, z) = g_{d-1}(t)\, z^{d-1} + \cdots + g_1(t)\, z + g_0(t).$$

If $u : \mathbb{R} \to \mathbb{C}$ is smooth satisfying $Q(\frac{d}{dt})\, u = h$ where $h : \mathbb{R} \to \mathbb{C}$ is continuous, then

$$u(t) = \sum_{j=0}^{d-1} u^{(j)}(0)\, g_j(t) + \int_0^t g_{d-1}(t - y)\, h(y)\, dy.$$

Assume that each a_p is an eigenvalue, let $A = S + N$ (S semisimple, N nilpotent) be the Jordan decomposition of A, and $A = \sum A_p = \sum(a_p E_p + N_p)$ be its spectral decomposition. Then

$$E_p = \sum_{q=0}^{\alpha_p} b_{p,q}\, (A - a_p)^q\, Q_p(A), \quad N_p = \sum_{q=0}^{\alpha_p-1} b_{p,q}\, (A - a_p)^{q+1}\, Q_p(A).$$

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